SEMINAR: ARITHMETIC STATISTIC (SOSE 2023)

DR. VICTORIA CANTORAL

1. Important information

Evaluation: Presentation and Handout
Time & Place: Fridays 16:16 - 17:45, 1101 - B302 Klaus Fröhlich Hörsaal
First meeting: 14.04.2023

2. Description

Consider an elliptic curve E defined over the rational numbers given by its minimal Weierstrass equation $E : y^2 = x^3 + ax + b$. Let p be a prime number not dividing the discriminant of the elliptic curve. Let us define the following quantity

$$N_E(P) := |E(\mathbb{F}_p)|.$$

In 1933 Hasse proved Artin's conjecture which asserts that $|N_E(p) - p| \leq 2\sqrt{p}$.

Question 2.1. *What is the distribution of* $\frac{N_E(p)-p}{\sqrt{p}}$ *in the interval* $[-2, 2]$ *when* $p \to \infty$, *is it equidistributed?*

The former has fascinated several mathematicians and laid the ground for the famous Sato–Tate conjecture. This conjecture is closely related to the famous Riemann hypothesis. It predicts the distribution of the zeros of the Riemann zêta function. More precisely, it asserts that for any elliptic curve without complex multiplication, defined over a number field K, the traces of the Frobenius are equidistributed. This conjecture is still open and only a few cases are known.

The first aim of this seminar is to introduce the Sato–Tate conjecture and give an overview of the known results. Subsequently, we will study several generalizations of this conjecture. For instance, one could generalize this conjecture to higher-dimension abelian varieties, $K3$ surfaces, and pure motives of odd weight.

For further details see the description in Stud.IP and feel free to contact me if you would like to give a talk.

3. Description of the talks

Apr. 28th: - Elliptic curves, abelian varieties, Albert's classification of abelian varieties. Frobenius automorphism, Hasse theorem (conjecture of Artin), and Deligne theorem (conjecture of Weil for abelian varieties).
[Mum70, HS00]

May 5th: - ℓ-adic representations associated to elliptic curves or abelian varieties, Frobenius traces, and L-functions associated to an elliptic curve or to an ℓ-adic representation. Euler product of L-functions and first properties.
[Ser68, Chap. 1, §2.3, §2.5]

Date: April 7, 2023.

May 12th: - Equidistribution statements for L-functions.
[Ser68, Appendix I-18], [Fit15, §2], [Sut19, §2.1, §2.2, §2.3], and [Ser12, Chap. 8]

May 19th: - Proof of the Sato-Tate conjecture for CM-elliptic curves. Results of Hecke and Deuring and examples.
[Sut19, §2.4], [Ser12, Chap. 8]

May 26th: - Statement of the Sato-Tate conjecture for elliptic curves without CM. Known results and description of the techniques that have been used.
[Sut19, §2.5], [LG15], and [Ser12, Chap. 8]

Jun. 2nd: - Statement of the generalized Sato-Tate conjecture for abelian varieties and known results.
[LG15], [Ser12, Chap. 8]

Jun. 9th: - Classification of the Sato-Tate groups for abelian surfaces.
[FKRS12]

Jun. 16th: - Classification of the Sato-Tate groups for abelian threefolds.
[FKS21]

Jun. 23th: - Variants of the Sato-Tate conjecture and Lang-Trotter conjecture.
[Jam16]

Jun. 30th: - Effective Sato-Tate conjecture for abelian varieties and applications.
[BFK20]

Jul. 7th: - Determining monodromy groups of abelian varieties.
[Zyw20]

Jul. 14th: - Geometry of $K3$ surfaces: definition, examples, and properties.
[VA17, §1, §2, §3]

Jul. 21th: - Statement of the Sato-Tate conjecture for $K3$ surfaces.
[EJ21]

References

[BFK20] Alina Bucur, Francesc Fité, and Kiran S. Kedlaya, *Effective Sato-Tate conjecture for abelian varieties and applications*, arXiv, 2020. ↑3

[EJ21] Andreas-Stephan Elsenhans and Jörg Jahnel, *Frobenius trace distributions for K3 surfaces*, arXiv, 2021. ↑3

[Fit15] Francesc Fité, *Equidistributions, L-Functions, and Sato–Tate Groups*, Contemporary Mathematics **649** (2015). ↑3

[FKS21] Francesc Fité, Kiran S. Kedlaya, and Andrew V. Sutherland, *Sato-Tate groups of abelian threefolds: a preview of the classification*, Arithmetic, geometry, cryptography and coding theory, 2021, pp. 103–129, DOI 10.1090/conm/770/15432. ↑3

[FKRS12] Francesc Fité, Kiran S. Kedlaya, Víctor Rotger, and Andrew V. Sutherland, *Sato-Tate distributions and Galois endomorphism modules in genus 2*, Compos. Math. **148** (2012), no. 5, 1390–1442, DOI 10.1112/S0010437X12000279. ↑3

[HS00] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000. An introduction. ↑3

[Jam16] Kevin James, *Variants of the Sato-Tate and Lang-Trotter conjectures*, Frobenius distributions: Lang-Trotter and Sato-Tate conjectures, 2016, pp. 175–184, DOI 10.1090/conm/663/13354. ↑3

[LG15] Elisa Lorenzo García, *On the Sato-Tate conjecture*, 2015. ↑3

[Mum70] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1970. ↑3

[Ser68] Jean-Pierre Serre, *Abelian ℓ-adic representation and elliptic curves*, 1968. ↑3

[Ser12] ———, *Lectures on $N_X(p)$*, Vol. 11, Chapman & Hall/CRC Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2012. ↑3

[Sut19] Andrew V. Sutherland, *Sato-Tate distributions*, Analytic methods in arithmetic geometry, 2019, pp. 197–248, DOI 10.1090/conm/740/14904. ↑3

[VA17] Anthony Várilly-Alvarado, *Arithmetic of K3 surfaces*, Geometry over nonclosed fields, 2017, pp. 197–248. ↑3

[Zyw20] David Zywina, *Determining monodromy groups of abelian varieties*, arXiv, 2020. ↑3